
Chapter 12.1: Limits of Sequences

Definition: A **sequence** in a set S is a function from $\mathbb{N}$ to S.

Definition (Limit of a sequence):
If, $\forall\ \varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $\forall n > N$, $\mid x_n - x \mid \leq \varepsilon$, then a sequence (x_n) of real numbers **converges** to the real number x.

(We write $\lim_{n\to\infty} x_n = x$, "$x$" is the limit of the sequence (x_n).)

Definition: If a sequence (x_n) does not converge to some real number, then the sequence (x_n) diverges.

Write the negation of convergence using quantifiers.

Examples

1. Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. Prove that $\lim_{n\to\infty} 1 = 1$.

3. Prove that $\lim_{n\to\infty} \frac{3}{2n+1} = 0$.

4. Prove that $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

5. Prove that the sequence $a_n = 1 + (-1)^n$ is divergent.

Examples

1. Prove that $\lim_{n\to\infty} \frac{n-2}{2n+1} = \frac{1}{2}$.

2. Prove that $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$.

3. Prove that $\lim_{n\to\infty} \frac{2n}{n^2+3} = 0$.

4. Prove that $\lim_{n\to\infty} \frac{2n}{n^2-3} = 0$.

5. Prove that $\lim_{n\to\infty} \frac{n^2+2n}{n^3-5} = 0$.

Some Properties of Real Numbers
Prove the following.

Proposition. Let $x, y \in \mathbb{R}$. Then $x = y$ if and only if $\forall \varepsilon > 0$ we have $|x - y| \leq \varepsilon$.

Some properties of limit.

Theorem 1. If a sequence (a_n) converges, then its limit is unique.

Theorem 2. Every convergent sequence must be bounded.

Theorem 3. Algebraic rules for sequences:
Let $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$.

(a) For $k \in \mathbb{R}$, $\lim_{n\to\infty} ks_n = k \lim_{n\to\infty} s_n = ks$.

(b) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

(c) $\lim_{n\to\infty} (s_n \cdot t_n) = s \cdot t$.

(d) For all n, $s_n \neq 0$ and $s \neq 0$, $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ and $\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s}$.

Divergence

Definition

(1) If $\forall M > 0$, $\exists N$ such that $\forall n > N$, $n \in \mathbb{N}$, $s_n > M$,
then the sequence diverges to $+\infty$. We write $\lim_{n\to\infty} s_n = +\infty$.

(2) If $\forall M < 0$, $\exists N$ such that $\forall n > N$, $n \in \mathbb{N}$, $s_n < M$,
then the sequence diverges to $-\infty$. We write $\lim_{n\to\infty} s_n = -\infty$.

Examples

1. Give a formal proof that $\lim_{n\to\infty} (\sqrt{n} + 7) = +\infty$.

2. Prove that $\lim_{n\to\infty} \dfrac{n^2+4}{n+2} = +\infty$.

3. Prove that $\lim_{n\to\infty} \dfrac{n^3}{1-n} = -\infty$.